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BOUNDEDNESS AND COMPACTNESS OF SOME TOEPLITZ OPERATORS

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Abstract. We consider the problem to determine when a Toeplitz operator is bounded on weighted Bergman spaces. We introduce some set CG of symbols and we prove that Toeplitz operators induced by elements of CG are bounded and characterize when Toeplitz operators are compact and show that each element of CG is related with a Carleson measure.

1. Introduction

Let dA denote normalized Lebesgue area measure on the unit disk $\mathbb{D}$. For $\alpha > -1$, the weighted Bergman space A^p_α consists of the analytic functions in $L^p(\mathbb{D}, dA_\alpha)$, where $dA_\alpha(z) = (\alpha+1)(1-|z|^2)^\alpha dA(z)$. Since A^2_α is a closed subspace of $L^2(\mathbb{D}, dA_\alpha)$, for any $z \in \mathbb{D}$, there is a unique function K^α_z in A^2_α such that $f(z) =< f, K^\alpha_z >$ for all $f \in A^2_\alpha$, in fact, $K^\alpha_z(w) = \dfrac{1}{(1-\overline{z}w)^{2+\alpha}}$ and the normalized reproducing kernel k^α_z is the function $\dfrac{K^\alpha_z(w)}{||K^\alpha_z||_{2,\alpha}} = \dfrac{(1-|z|^2)^{1+\frac{\alpha}{2}}}{(1-\overline{z}w)^{2+\alpha}}$, where the norm $||\cdot||_{p,\alpha}$ and the inner product are taken in the space $L^p(\mathbb{D}, dA_\alpha)$ and $L^2(\mathbb{D}, dA_\alpha)$, respectively.

For a linear operator S on A^2_α, S induces a functions $\widetilde{S}$ on $\mathbb{D}$ given by $\widetilde{S}(z) =< Sk^\alpha_z, k^\alpha_z >$, $z \in \mathbb{D}$. The function $\widetilde{S}$ is called the Berezin transform of S.

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For $u \in L^1(\mathbb{D}, dA_\alpha)$, the Toeplitz operator T_u^α with symbol u is the operator on A_α^2 defined by $T_u^\alpha(f) = P_\alpha(uf)$, $f \in A_\alpha^2$, where P_α is the orthogonal projection from $L^2(\mathbb{D}, dA_\alpha)$ onto A_α^2 and let $\widetilde{u}$ denote $\widetilde{T_u^\alpha}$. Many mathematicians working in operator theory are interested in the boundedness and compactness of Toeplitz operators on the Bergman spaces. It is well-known that the Toeplitz operator T_u^α induced by any element of $L^\infty(\mathbb{D}, dA_\alpha)$ is bounded. Since $L^\infty(\mathbb{D}, dA_\alpha)$ is dense in $L^1(\mathbb{D}, dA_\alpha)$, for any $u \in L^1(\mathbb{D}, dA_\alpha)$, T_u^α is densely defined on A_α^2 but in general, T_u^α is not bounded. We note that Berezin transforms and Carleson measures are useful tools in the syudy of Toeplitz operators ([2], [4], [5]). Using those tools, many mathematicians working in the operator theory characterized the boundedness and compactness of Toeplitz operators.

In this paper, we introduce some set CG and prove that Toeplitz operators induced by elements of CG are bounded and $||u||_G$ having vanishing property implies the compactness of Toeplitz operators T_u^α and $T_{\overline{u}}^\alpha$.

Sections 3 contains some upper bounds of Toeplitz operators induced by elements of CG and relationship between elements of CG and Carleson measures and we deal with the compactness of appropriate products of Toeplitz operators and Hankel operators.

Throughout this paper, we use the symbol $A \preceq B$ for nonnegative constants A and B to indicate that A is dominated by B time some positive constant and p' to denote the conjugate of p, that is, $\frac{1}{p} + \frac{1}{p'} = 1$.

2. Some linear operators

A nice survey of previously known results connecting Toeplitz operators with bounded symbol can be found in [2].

For $z \in \mathbb{D}$, let $\varphi_z(w) = \dfrac{z-w}{1-\overline{z}w}$. Then φ_z is an element of $Aut(\mathbb{D})$ which is the set of all bianalytic map of $\mathbb{D}$ onto $\mathbb{D}$. Moreover, $\varphi_z \circ \varphi_z$ is the identity map on $\mathbb{D}$ and $Aut(\mathbb{D})$ is the Möbius group under composition.

For $\alpha > -1$ and $z \in \mathbb{D}$, let $U_z^\alpha : L^2(\mathbb{D}, dA_\alpha) \to L^2(\mathbb{D}, dA_\alpha)$ be an isometry operator defined by

$$U_z^\alpha f(w) = f \circ \varphi_z(w) \frac{\left(1-|z|^2\right)^{1+\frac{\alpha}{2}}}{\left(1-\overline{z}w\right)^{2+\alpha}},$$

$f \in L^2(\mathbb{D}, dA_\alpha)$ and $w \in \mathbb{D}$.

Since $(1-\overline{z}\varphi_z(w))^{2+\alpha}=\Big(\dfrac{1-|z|^2}{1-\overline{z}w}\Big)^{2+\alpha}$, $(U_z^\alpha)^{-1}=U_z^\alpha$ and hence U_z^α is a self-adjoint unitary operator on A_α^2 and $U_z^\alpha 1=k_z^\alpha(w)$.

For a linear operator S on A_α^2, define S_z by $U_z^\alpha S U_z^\alpha$. Since U_z^α is a self-inverse operator, S_z is the operator given by conjugation with U_z^α.

Now we are ready to state useful properties.

Lemma 2.1. *For $u\in L^1(\mathbb{D},dA_\alpha)$ and $z\in\mathbb{D}$, $(T_u^\alpha)_z=T_{u\circ\varphi_z}^\alpha$.*

Proof. Take any f in A_α^2 and any w in $\mathbb{D}$. Since U_z^α is self-adjiont,

$$\begin{aligned}
U_z^\alpha T_u^\alpha(f)(w) &= < U_z^\alpha T_u^\alpha(f), K_w^\alpha > \\
&= < U_z^\alpha(uf), K_w^\alpha > \\
&= < (u\circ\varphi_z)(f\circ\varphi_z)\frac{(1-|z|^2)^{1+\frac{\alpha}{2}}}{(1-\overline{z}w)^{2+\alpha}}, K_w^\alpha > \\
&= < T_{u\circ\varphi_z}^\alpha(U_z^\alpha f), K_w^\alpha > \\
&= T_{u\circ\varphi_z}^\alpha(U_z^\alpha f)(w).
\end{aligned}$$

Thus $(T_u^\alpha)_z=T_{u\circ\varphi_z}^\alpha$. □

Corollary 2.2. *For $u_1,u_2,\cdots,u_n\in L^1(\mathbb{D},dA_\alpha)$ and $z\in\mathbb{D}$,*

$$U_z^\alpha T_{u_1}^\alpha T_{u_2}^\alpha\cdots T_{u_n}^\alpha U_z^\alpha=T_{u_1\circ\varphi_z}^\alpha\cdots T_{u_n\circ\varphi_z}^\alpha.$$

Proof. If follows immediately from the fact that $(U_z^\alpha)^{-1}=U_z^\alpha$ and Lemma 2.1. □

Proposition 2.3. *For $u\in L^1(\mathbb{D},dA_\alpha)$ and $z\in\mathbb{D}$, $\widetilde{T_{u\circ\varphi_z}^\alpha}=\widetilde{T_u^\alpha}\circ\varphi_z$ and hence $(\widetilde{T_u^\alpha})_z=\widetilde{T_{u\circ\varphi_z}^\alpha}=\widetilde{T_u^\alpha}\circ\varphi_z$.*

Proof. Take any w in $\mathbb{D}$. Since $< u\circ\varphi_z k_w^\alpha, k_w^\alpha > = < uk_{\varphi_z(w)}^\alpha, k_{\varphi_z(w)}^\alpha >$,

$$\begin{aligned}
\widetilde{T_{u\circ\varphi_z}^\alpha}(w) &= < T_{u\circ\varphi_z}^\alpha k_w^\alpha, k_w^\alpha > \\
&= < u\circ\varphi_z k_w^\alpha, k_w^\alpha > \\
&= < uk_{\varphi_z(w)}^\alpha, k_{\varphi_z(w)}^\alpha > \\
&= < P_\alpha(uk_{\varphi_z(w)}^\alpha), k_{\varphi_z(w)}^\alpha > \\
&= \widetilde{T_u^\alpha}(\varphi_z(w)) \\
&= \widetilde{T_u^\alpha}\circ\varphi_z(w).
\end{aligned}$$

This completes the proof. □

Proposition 2.4. *If $S: A_\alpha^2\to A_\alpha^2$ is a bounded linear operator then $\widetilde{S}$ and $S_z 1$ are in $L^2(\mathbb{D},dA_\alpha)$.*

Proof. Since $||S_z 1||_{2,\alpha} = ||SU_z^\alpha 1||_{2,\alpha} \le ||S||$ and

$$||\widetilde{S}||_{2,\alpha} = \int_{\mathbb{D}} |\widetilde{S}(z)|^2 dA_\alpha(z) \le \int_{\mathbb{D}} ||S||^2 dA_\alpha(z) = ||S||^2,$$

$\widetilde{S}$ and $S_z 1$ are in $L^2(\mathbb{D}, dA_\alpha)$. $\square$

We notice that $P_\alpha : L^2(\mathbb{D}, dA_\alpha) \to A_\alpha^2$ is bounded linear operator and hence for any $u \in L^\infty(\mathbb{D}, dA_\alpha)$, $||P_\alpha(uf)||_{2,\alpha} \le ||u||_\infty ||f||_{2,\alpha}$. Thus T_u^α is a bounded linear operator. Moreover, we extend the domain of P_α to $L^1(\mathbb{D}, dA_\alpha)$ and for $f \in A_\alpha^1$ and $z \in \mathbb{D}$, $f(z) = \int_{\mathbb{D}} f(w)\overline{K_z^\alpha(w)} dA_\alpha(w)$.

We define $f(z) = \sum_{k=1}^{\infty} k\chi_{(\frac{1}{2^k} - \frac{1}{2^{k+1}}, \frac{1}{2^k})}(|z|)$ for all $z \in \mathbb{D}$. Then f is a radial function and $f \notin L^\infty(\mathbb{D}, dA_\alpha)$. Since

$$\begin{aligned} ||f||_{1,\alpha} &= \int_{\mathbb{D}} |f(z)|(1-|z|^2)^\alpha dA(z) \\ &\le \begin{cases} (1-\frac{1}{4})^\alpha \sum_{k=1}^{\infty} \frac{k}{2^{k+1}} & ,\alpha < 0 \\ \sum_{k=1}^{\infty} \frac{k}{2^{k+1}} & ,\alpha \ge 0 \end{cases} , f \in L^1(\mathbb{D}, dA_\alpha). \end{aligned}$$

For $p > 2$,

$$||(T_f^\alpha)_z 1||_{p,\alpha} = ||U_z^\alpha T_f^\alpha U_z^\alpha 1||_{p,\alpha} \le ||fk_z^\alpha||_{p,\alpha} < \infty$$

because $\sup\{|k_z^\alpha(w)| : |w| \le \frac{1}{2}\} \le 2^{2+\alpha}$. Since for each $z \in \mathbb{D}$,

$$\widetilde{|f|}(z) = \int_{\mathbb{D}} |k_z^\alpha(w)|^2 |f(w)| dA_\alpha(w) \le 2^{4+2\alpha} c \sum_{k=1}^{\infty} \frac{k}{2^{k+1}}$$

for some constant c, $|f|dA_\alpha$ is a Carleson measure and hence T_f^α is a bounded linear operator. But every element of $L^1(\mathbb{D}, dA_\alpha)$ does not imply a bounded Toeplitz operator. Let $CG = \{u \in L^1(\mathbb{D}, dA_\alpha) : \sup_z ||(T_u^\alpha)_z 1||_{p,\alpha} < \infty$ and $\sup_z ||(T_u^{\alpha *})_z 1||_{p,\alpha} < \infty$ for some $p \in (2, \infty)\}$. Suppose $f, g \in A_\alpha^2$. Since $< T_u^\alpha f, g >=< uf, g >=< f, \overline{u} g >=< f, T_{\overline{u}}^\alpha g >$, $(T_u^\alpha)^* = T_{\overline{u}}^\alpha$. If $||(T_u)_z 1||_{p,\alpha} < \infty$ then $||(T_u^{\alpha *})_z 1||_{p,\alpha} = ||(T_{\overline{u}}^\alpha)_z 1||_{p,\alpha} < \infty$ and clearly CG is closed under the formation of conjugation and hence $\{T_u^\alpha : u \in CG\}$ is self-adjoint in $\mathcal{L}(A_\alpha^2)$ which is the set of all bounded linear operators on A_α^2. Moreover, CG is a vector space over $\mathbb{C}$ and we definde $||u||_G = \max\{\sup_z ||(T_u^\alpha)_z 1||_{p,\alpha}, \sup_z ||(T_{\overline{u}}^\alpha)_z 1||_{p,\alpha}\}$.

By the above observation, $L^\infty(\mathbb{D}, dA_\alpha)$ is a proper subset of CG. Since $f(z) = 0$ for all $|z| > \frac{1}{2}$, $\lim_{z\to\partial\mathbb{D}} \widetilde{T_f^\alpha}(z)=0= \lim_{z\to\partial\mathbb{D}} ||(T_f^\alpha)_z 1||_{p,\alpha}$. Since $T_f^\alpha(z^n) \neq 0$ for all $n \in \mathbb{N}$, T_f^α has an infinite-dimensional range and hence it is not compact, that is, the vanishing property does not imply the compactness of Toeplitz operators.

3. Some operators

This section contains the boundedness of some operators. We begin by starting well-known lemma (see Lemma 3.10 in [5]) which is some integral estimates.

Lemma 3.1. *Suppose $a - 1 < \alpha$. If $a + b < 2 + \alpha$ then* $\int_{\mathbb{D}} \frac{dA_\alpha(w)}{(1-|w|^2)^a |1-\overline{z}w|^b}$ *is bounded on $\mathbb{D}$.*

Note that $(T_u^\alpha)^* = T_{\overline{u}}^\alpha$. Thus for $z \in \mathbb{D}$,

$$(T_u^\alpha)^* K_w^\alpha(z) =< (T_u^\alpha)^* K_w^\alpha, K_z^\alpha >=< K_w^\alpha, T_u^\alpha K_z^\alpha >= \overline{T_u^\alpha K_z^\alpha(w)}.$$

Moreover, $||(T_u^\alpha)_z 1||_{t,\alpha}$ in the right side of the next lemma may not be finite but it will be infinite, making the corresponding inequality true.

Lemma 3.2. *Suppose $u \in L^1(\mathbb{D}, dA_\alpha)$ and $0 < a < 1$. If $2 < \frac{2+\alpha}{a} < t$ then there is a constant c such that*

$$\int_{\mathbb{D}} \frac{|(T_u^\alpha K_z^\alpha)(w)|}{(1-|w|^2)^a} dA_\alpha(w) \quad \leq \quad \frac{c||(T_u^\alpha)_z 1||_{t,\alpha}}{(1-|z|^2)^a}$$

for all $z \in \mathbb{D}$ and

$$\int_{\mathbb{D}} \frac{|(T_u^\alpha K_z^\alpha)(w)|}{(1-|z|^2)^a} dA_\alpha(z) \quad \leq \quad \frac{c||(T_{\overline{u}}^\alpha)_w 1||_{t,\alpha}}{(1-|w|^2)^a}$$

for all $w \in \mathbb{D}$.

Proof. Take any z in $\mathbb{D}$. Since $U_z^\alpha 1 = k_z^\alpha$, $T_u^\alpha K_z^\alpha = \frac{(T_u^\alpha)_z 1 \circ \varphi_z (\varphi_z')^{1+\frac{\alpha}{2}}}{(1-|z|^2)^{1+\frac{\alpha}{2}}}$ and hence put $w = \varphi_z(\lambda)$ to obtain the following :

$$\begin{aligned}
&\int_{\mathbb{D}} \frac{|T_u^\alpha K_z^\alpha(w)|}{(1-|w|^2)^a} dA_\alpha(w) \\
&= \int_{\mathbb{D}} \frac{|(T_u^\alpha)_z 1(\lambda)||\varphi_z'(\varphi_z(\lambda))|^{1+\frac{\alpha}{2}}}{(1-|z|^2)^{1+\frac{\alpha}{2}}(1-|\varphi_z(\lambda)|^2)^a} |\varphi_z'(\lambda)|^2 (1-|\varphi_z(\lambda)|^2)^\alpha dA(\lambda) \\
&= \frac{1}{(1-|z|^2)^a} \int_{\mathbb{D}} \frac{|(T_u^\alpha)_z 1(\lambda)|}{|1-\overline{z}\lambda|^{2-2a+\alpha}(1-|\lambda|^2)^{a-\alpha}} dA(\lambda) \\
&\leq \frac{||(T_u^\alpha)_z 1||_{t,\alpha}}{(1-|z|^2)^a} \Big(\int_{\mathbb{D}} \frac{dA_\alpha(\lambda)}{(1-|\lambda|^2)^{at'}|1-\overline{z}\lambda|^{(2-2a+\alpha)t'}} \Big)^{\frac{1}{t'}}.
\end{aligned}$$

Here, the inequality comes from Hölder's inequality.
If $(2-a+\alpha)t' - \alpha < 2$ then the final integral is finite. Since $\frac{2+\alpha}{a} < t$, $t' < \frac{2+\alpha}{2-a+\alpha}$. This makes the corresponding inequality true. The second inequality follows from the above observation. □

Corollary 3.3. *Suppose $0 < a < 1$ and $||u||_G$ is finite with respect to $||\cdot||_{p,\alpha}$ for some $p \in (2,\infty)$, that is, $u \in CG$. If $2 < \frac{2+\alpha}{a} < p$ then there is a constant c such that*

$$\int_{\mathbb{D}} \frac{|(T_u^\alpha K_z^\alpha)(w)|}{(1-|w|^2)^a} dA_\alpha(w) \leq \frac{c||(T_u^\alpha)_z||_{p,\alpha}}{(1-|z|^2)^a} \preceq \frac{||u||_G}{(1-|z|^2)^a}$$

for all $z \in \mathbb{D}$ and

$$\int_{\mathbb{D}} \frac{|(T_u^\alpha K_z^\alpha)(w)|}{(1-|z|^2)^a} dA_\alpha(z) \leq \frac{c||(T_u^\alpha)_w||_{p,\alpha}}{(1-|w|^2)^a} \preceq \frac{||u||_G}{(1-|w|^2)^a}$$

for all $w \in \mathbb{D}$.

Proof. If follows immediately from the definition of $||u||_G$ and Lemma 3.2. □

Proposition 3.4. *If $u \in CG$ and $||u||_G$ is finite with respect to $||\cdot||_{t,\alpha}$ then $|T_{\overline{u}}^\alpha(h)(w)| \leq \dfrac{1}{(1-|w|^2)^{1+\frac{\alpha}{2}}}||h||_{2,\alpha}||u||_{t,\alpha}$ for every $h \in A_\alpha^2$ and every $w \in \mathbb{D}$.*

Proof. Suppose $h \in A_\alpha^2$ and $w \in \mathbb{D}$. Then

$$(T_{\overline{u}}^\alpha h)(w) = < T_{\overline{u}}^\alpha h, K_w^\alpha > = \frac{1}{(1-|w|^2)^{1+\frac{\alpha}{2}}} \times < h, \overline{u}k_w^\alpha > .$$

By Hölder's inequality, we get
$| < h, \overline{u}k_w^\alpha > | \leq ||h||_{t',\alpha}||\overline{u}k_w^\alpha||_{t,\alpha}$. Since $1 < t' < 2$ and $A_\alpha(\mathbb{D}) = 1$, $||h||_{t',\alpha} \leq ||h||_{t,\alpha}$ and hence one has the result. □

Suppose $f \in A^2_\alpha$ and $z \in \mathbb{D}$. Then

$$\begin{aligned}(T^\alpha_u f)(z) &= < T^\alpha_u f, K^\alpha_z > \\ &= \int_{\mathbb{D}} f(w)\overline{((T^\alpha_u)^* K^\alpha_z)(w)} dA_\alpha(w) \\ &= \int_{\mathbb{D}} f(w) T^\alpha_u K^\alpha_w(z) dA_\alpha(w).\end{aligned}$$

Thus T^α_u is the integral operator with kernel $T^\alpha_u K^\alpha_w(z)$ and hence we find some upper bound of $||T^\alpha_u||_p$ to use the Schur test (see page 126 of [3]), where $||T^\alpha_u||_p$ is the operator norm on A^p_α.

Theorem 3.5. *Suppose $u \in CG$ and $||u||_G$ is finite with respect to $||\cdot||_{p,\alpha}$. If $pp'(2+\alpha) < t$ then T^α_u is a bounded linear operator on A^p_α and $A^{p'}_\alpha$ and $||T^\alpha_u||_p \preceq ||u||_G$.*

Proof. Since $0 < \frac{1}{pp'} < 1$, let $h(\lambda) = \frac{1}{(1-|\lambda|^2)^{pp'}}$. Then h is a positive measurable function. Since $||(T^\alpha_u)_z 1||_{t,\alpha}$ and $||(T^\alpha_{\overline{u}})_z 1||_{t,\alpha}$ are less than or equal to $||u||_G$, the results follow from Lemma 3.2 and the Schur test. □

Using the concept of a Carleson measure, we get the boundness and compactness of Toeplitz operators.

Proposition 3.6. *Suppose $u \in CG$ and $||u||_G$ is finite with respect to $||\cdot||_{t,\alpha}$.*

(1) *Then $|u|dA_\alpha$ is a Carleson measure on A^p_α and hence T^α_u is a bounded linear operator.*
(2) *If $||(T^\alpha_u)_z 1||_{t,\alpha} \to 0$ as $z \to \partial\mathbb{D}$ then T^α_u is compact.*

Proof. (1) For $z \in \mathbb{D}$,
$$\begin{aligned}|\widetilde{u}(z)| &= | < T^\alpha_u k^\alpha_z, k^\alpha_z > | \\ &= (1-|z|^2)^{1+\frac{\alpha}{2}} | < T^\alpha_u K^\alpha_z, k^\alpha_z > | \\ &\le (1-|z|^2)^{1+\frac{\alpha}{2}} ||T^\alpha_u K^\alpha_z||_{2,\alpha} \\ &= (1-|z|^2)^{1+\frac{\alpha}{2}} ||(T^\alpha_u)_z 1||_{2,\alpha} \\ &\le (1-|z|^2)^{1+\frac{\alpha}{2}} ||(T^\alpha_u)_z 1||_{t,\alpha},\end{aligned}$$
where the last inequality follows from $A_\alpha(\mathbb{D}) = 1$.
Since $\widetilde{u}$ is bounded, $|u|dA_\alpha$ is a Carleson measure on A^p_α.

(2) In the proof of (1), for $z \in \mathbb{D}$, $|\widetilde{u}(z)| \le (1-|z|^2)^{1+\frac{\alpha}{2}} ||(T^\alpha_u)_z 1||_{t,\alpha}$ and hence $|u|dA_\alpha$ is a vanishing Carleson measure. Thus T^α_u is a compact linear operator. □

COROLLARY 3.7. *Suppose $u \in CG$ and $||u||_G$ is finite with respect to $||\cdot||_{p,\alpha}$. If $||u||_G$ vanishes on $\partial\mathbb{D}$ then T_u^α and $T_{\overline{u}}^\alpha$ are compact opeators.*

Proof. It follows immediately from the fact that $||(T_u^\alpha)_z 1||_{t,\alpha}$ and $||(T_{\overline{u}}^\alpha)_z 1||_{t,\alpha}$ are less than or equal to $||u||_G$. □

PROPOSITION 3.8. *Suppose $u \in CG$ and $||u||_G$ is finite with respect to $||\cdot||_{t,\alpha}$. If T_u^α is a compact operator then $||(T_u^\alpha)_z 1||_{2,\alpha} \to 0$ as $z \to \partial\mathbb{D}$ and hence $\widetilde{u}$ has the vanishing property on $\partial\mathbb{D}$. Moreover, $||(T_u^\alpha)_z 1||_{t,\alpha} \to 0$ as $z \to \partial\mathbb{D}$.*

Proof. We note that H^∞ is dense in A_α^2. Take any f in A_α^2. Then $< f, k_z^\alpha > = (1-|z|^2)^{1+\frac{\alpha}{2}} f(z)$ and hence $k_z^\alpha \to 0$ weakly in A_α^2 as $z \to \partial\mathbb{D}$. Since $||(T_u^\alpha)_z 1||_{2,\alpha} = ||T_u^\alpha k_z^\alpha||_{2,\alpha}$ and T_u^α is compact, $||(T_u^\alpha)_z 1||_{2,\alpha} \to 0$ as $z \to \partial\mathbb{D}$. □

For $u \in L^1(\mathbb{D}, dA_\alpha)$, we define an operator $H_u^\alpha : A_\alpha^2 \to (A_\alpha^2)^\perp$ by $H_u^\alpha(g) = (I - P_\alpha)(ug)$, $g \in A_\alpha^2$. Then H_u^α is called the Hankel operator on the weighted Bergman space with symbol u. Since $L^\infty(\mathbb{D}, dA_\alpha)$ is dense in $L^1(\mathbb{D}, dA_\alpha)$, H_u^α is densely defined and if $u \in L^\infty(\mathbb{D}, dA_\alpha)$ then $||H_u^\alpha|| \le ||u||_\infty$ and hence H_u^α is bounded. By Lemma 2.1, $(T_u^\alpha)_z = T_{u\circ\varphi_z}^\alpha$ and hence $||(H_u^\alpha)_z 1||_{2,\alpha} = ||H_u^\alpha k_z^\alpha||_{2,\alpha} \le ||H_u^\alpha||$ and $(H_u^\alpha)_z = (I - T_u^\alpha)_z = I - T_{u\circ\varphi_z}^\alpha = H_{u\circ\varphi_z}^\alpha$. Thus one has the following properties :

PROPOSITION 3.9. *Suppose $u_1, u_2 \in L^1(\mathbb{D}, dA_\alpha)$ and $u_1 = u_2 \circ \varphi_z$ for some $z \in \mathbb{D}$. Then the following pairs are unitary equivalent :*
(1) $T_{u_1}^\alpha$ and $T_{u_2}^\alpha$ *(2) $H_{u_1}^\alpha$ and $H_{u_2}^\alpha$.*

PROPOSITION 3.10. *Suppose H_u^α is bounded, where $u \in L^1(\mathbb{D}, dA_\alpha)$. Then $(H_u^\alpha)_z 1$ and $H_u^\alpha k_z^\alpha$ are in $L^2(\mathbb{D}, dA_\alpha)$ and $H_{u\circ\varphi_z}^\alpha$ is bounded.*

Proof. By the above observation, $(H_u^\alpha)_z 1$ and $H_u^\alpha k_z^\alpha$ are in $L^2(\mathbb{D}, dA_\alpha)$. Take any f in A_α^2. Since $(H_u^\alpha)_z = H_{u\circ\varphi_z}^\alpha$, $||H_{u\circ\varphi_z}^\alpha(f)||_{2,\alpha} = ||(H_u^\alpha)_z f||_{2,\alpha}$ $= ||H_u^\alpha U_z^\alpha(f)||_{2,\alpha} \le ||H_u^\alpha|| ||f||_{2,\alpha}$. This completes the proof. □

PROPOSITION 3.11. *If $u^2 \in CG$ then H_u^α is bounded and hence we get the results of Proposition 3.10.*

Proof. Take any f in A_α^2. By Proposition 3.6, $|u|^2 dA_\alpha$ is a Carleson measure on A_α^2 and hence there is a constant c such that

$$\int_{\mathbb{D}} |f(z)|^2 |u(z)|^2 dA_\alpha(z) \le c||f||_{2,\alpha}^2.$$

Then $||H_u^\alpha(f)||_{2,\alpha}^2 = ||(I-P_\alpha)(uf)||_{2,\alpha}^2 \le ||uf||_{2,\alpha}^2 \le c||f||_{2,\alpha}^2$. Thus H_u^α is bounded. □

Consider some products of Toeplitz operators and Hankel operators. Suppos $u, v \in L^1(\mathbb{D}, dA_\alpha)$ and $f, g \in A_\alpha^2$. Since $< vf, P_\alpha(ug) > = < P_\alpha(vf), P_\alpha(ug) >$ and $< \overline{u}T_v^\alpha(f), g > = < T_{\overline{u}}^\alpha T_v^\alpha(f), g >$, $< (H_u^\alpha)^* H_v^\alpha(f), g > = < \overline{u}vf, g > - < T_v^\alpha(f), ug > - < vf, P_\alpha(ug) > + < T_{\overline{u}}^\alpha T_v^\alpha(f), g > = < (T_{\overline{u}v}^\alpha - T_{\overline{u}}^\alpha T_v^\alpha)(f), g >$ and hence $(H_u^\alpha)^* H_v^\alpha = T_{\overline{u}v}^\alpha - T_{\overline{u}}^\alpha T_v^\alpha$. In particular, if $u = v$ then $(H_u^\alpha)^* H_u^\alpha = T_{|u|^2}^\alpha - T_{\overline{u}}^\alpha T_u^\alpha$. If H_u^α is compact then $(H_u^\alpha)^* H_u^\alpha$ is compact. Proposition 3.8 implies that $((H_u^\alpha)^* H_u^\alpha)^\sim(z) \to 0$ as $z \to \partial\mathbb{D}$ and hence $||H_u k_z^\alpha||_{2,\alpha} \to 0$ as $z \to \partial\mathbb{D}$ because $||H_u^\alpha k_z^\alpha||_{2,\alpha}^2 = < H_u^\alpha k_z^\alpha, H_u^\alpha k_z^\alpha > = ((H_u^\alpha)^* H_u^\alpha)^\sim(z)$.
Suppose u, v, u^2, v^2 are in CG and T_u^α and H_u^α are compact. Then $(T_u^\alpha)^*$ and $(H_u^\alpha)^*$ are also compact. Since U_z^α is a bounded linear operator and $(T_u^\alpha)_z = U_z^\alpha T_u^\alpha U_z^\alpha$, the above equality implies that the following are compact :

(1) $T_u^\alpha T_v^\alpha$ (2) $T_u^\alpha T_{\overline{v}}^\alpha$ (3) $T_{\overline{u}}^\alpha T_u^\alpha$ (4) $(H_u^\alpha)^* H_v^\alpha$ (5) $H_u^\alpha (H_v^\alpha)^*$
(6) $T_{\overline{u}v}^\alpha$ (7) $T_{u\overline{v}}^\alpha$ (8) $T_{|u|^2}^\alpha$ (9) $H_u^\alpha T_u^\alpha$ (10) $H_u^\alpha T_{\overline{u}}^\alpha$
(11) $H_v^\alpha T_u^\alpha$ (12) $T_{u\circ\varphi_z}^\alpha T_{v\circ\varphi_z}^\alpha$ (13) $T_{\overline{u\circ\varphi_z}}^\alpha T_{v\circ\varphi_z}^\alpha$
(14) $H_{u\circ\varphi_z}^\alpha (H_v^\alpha)^*$ (15) $(H_u^\alpha)^* H_{v\circ\varphi_z}^\alpha$.

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